Probabilistic Methods in Harmonic Analysis

Lecture 2: Interpolation

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1. Real interpolation

2. Complex interpolation

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L^p spaces

Let (X, Σ, μ) be a measure space.
All functions $f, g, \ldots : X \to \mathbb{C}$ are assumed to be measurable. We set

$$[f] = \{g\colon X \to \mathbb{C} \mid f = g \ \mu\text{-a.e.}\}$$

and often write simply f in place of $[f]$.

Definition

For $0 < p < \infty$,

$$L^p(X, \mu) = \left\{ [f] \,\middle|\, \int_X |f|^p \,\mathrm{d}\mu < \infty \right\}.$$

Proposition

$L^p(X, \mu)$ with the quasi-norm $\|f\|_{L^p} = \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{1/p}$ is a quasi-Banach space. It is a Banach space if $p \geq 1$.

For $p = \infty$, $L^\infty(X, \mu) = \{[f] \mid \|f\|_{L^\infty} = \operatorname{ess\,sup} |f| < \infty\}$.

Distribution function

Definition

For $f\colon X \to \mathbb{C}$, the distribution function $d_f\colon [0,\infty) \to [0,\infty]$ is defined as

$$d_f(\alpha) = \mu\left(\{|f| > \alpha\}\right).$$

Lemma

For $p \in (0,\infty)$,

$$\|f\|_{L^p}^p = p\int_0^\infty \alpha^{p-1} d_f(\alpha)\,\mathrm{d}\alpha.$$

Proof.

We have

$$\|f\|_{L^p}^p = \int_X |f|^p \, \mathrm{d}\mu = \int_X \int_0^{|f|} p\alpha^{p-1} \, \mathrm{d}\alpha \mathrm{d}\mu$$

$$= p \int_0^\infty \alpha^{p-1} \int_X \chi_{\{|f|>\alpha\}} \, \mathrm{d}\mu \mathrm{d}\alpha = p \int_0^\infty \alpha^{p-1} d_f(\alpha) \, \mathrm{d}\alpha. \qquad \square$$

Remark

Let $\varphi \in \mathscr{C}([0,\infty);\mathbb{R}) \cap \mathscr{C}^1((0,\infty);\mathbb{R})$, $\varphi(0) = 0$, and $\varphi' \geq 0$. Then

$$\int_X \varphi(|f|) \, \mathrm{d}\mu = \int_0^\infty \varphi'(\alpha) d_f(\alpha) \, \mathrm{d}\alpha.$$

Weak L^p spaces

Definition

For $0 < p < \infty$,

$$L^{p,\infty}(X,\mu) = \{[f] \mid \|f\|_{L^{p,\infty}} < \infty\},$$

where

$$\begin{aligned}\|f\|_{L^{p,\infty}} &= \inf\left\{C > 0 \mid d_f(\alpha) \le (C/\alpha)^p \text{ for all } \alpha > 0\right\} \\ &= \sup\left\{\gamma d_f(\gamma)^{1/p} \mid \gamma > 0\right\}.\end{aligned}$$

For $p = \infty$, $L^{\infty,\infty}(X,\mu) = L^\infty(X,\mu)$.

Again, $L^{p,\infty}(X,\mu)$ is a quasi-Banach space. It is a Banach space if $p > 1$ (under an equivalent norm).

Lemma

$L^p(X, \mu) \subseteq L^{p,\infty}(X, \mu)$.

Proof.

One has

$$\alpha^p d_f(\alpha) \leq \int_{\{|f|>\alpha\}} |f|^p \, d\mu \leq \|f\|_{L^p}^p. \qquad \square$$

Example

$|x|^{-n/p} \notin L^p(\mathbb{R}^n)$, but $|x|^{-n/p} \in L^{p,\infty}(\mathbb{R}^n)$.

Proposition

For $0 < p < r < q \leq \infty$,

$$L^{p,\infty}(X,\mu) \cap L^{q,\infty}(X,\mu) \subseteq L^r(X,\mu).$$

Proof.

We treat the case $q < \infty$. (The case $q = \infty$ is easier.)

Let $f \in L^{p,\infty}(X,\mu) \cap L^{q,\infty}(X,\mu)$ and set $B = \left(\|f\|^q_{L^{q,\infty}} / \|f\|^p_{L^{p,\infty}}\right)^{1/(q-p)}$. Then

$$\begin{aligned}\|f\|^r_{L^r} &= r\int_0^\infty \alpha^{r-1} d_f(\alpha)\,\mathrm{d}\alpha \leq r\int_0^\infty \alpha^{r-1} \min\left(\frac{\|f\|^p_{L^{p,\infty}}}{\alpha^p}, \frac{\|f\|^q_{L^{q,\infty}}}{\alpha^q}\right)\mathrm{d}\alpha \\ &= r\int_0^B \alpha^{r-p-1}\|f\|^p_{L^{p,\infty}}\,\mathrm{d}\alpha + r\int_B^\infty \alpha^{r-q-1}\|f\|^q_{L^{q,\infty}}\,\mathrm{d}\alpha \\ &= \frac{r}{r-p}\|f\|^p_{L^{p,\infty}} B^{r-p} + \frac{r}{q-r}\|f\|^q_{L^{q,\infty}} B^{r-q} = C(p,q,r)\,\|f\|^{(1-\theta)r}_{L^{p,\infty}}\|f\|^{\theta r}_{L^{q,\infty}} < \infty,\end{aligned}$$

where $\theta = \frac{1/p - 1/r}{1/p - 1/q} \in (0,1)$. $\square$

The Marcinkiewicz interpolation theorem

Let T be an operator which is defined on a linear subspace of the space of measurable functions on (X, μ) and takes values in the measurable functions on (Y, ν).

- T is said to be quasilinear if $|T(f+g)| \leq K(|T(f)| + |T(g)|)$ and $T(\lambda f) = |\lambda|\,|T(f)|$ for some $K > 0$ and all f, g and all $\lambda \in \mathbb{C}$.
- T is said to be sublinear if it is quasilinear with $K = 1$.

Theorem

Let $0 < p_0 < p < p_1 \leq \infty$ *and let* T *be a quasilinear operator* (*with constant* $K > 0$) *defined on* $L^{p_0}(X, \mu) + L^{p_1}(X, \mu)$. *Suppose that*

$$\|T(f)\|_{L^{p_j,\infty}} \leq M_j \|f\|_{L^{p_j}}, \quad j = 0, 1,$$

for certain constants M_0, M_1. *Then*

$$\|T(f)\|_{L^p} \leq C(p, p_0, p_1, K) M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p},$$

where $\theta \in (0, 1)$ *is given by* $1/p = (1-\theta)/p_0 + \theta/p_1$.

Proof

We treat the case $p_1 < \infty$.

Let $f \in L^p(X,\mu)$ and $\alpha > 0$. Split f at height $\delta\alpha$ for some $\delta > 0$ to be chosen later, i.e., $f = f_0^\alpha + f_1^\alpha$, where

$$f_0^\alpha = \chi_{\{|f|>\delta\alpha\}} f, \quad f_1^\alpha = \chi_{\{|f|\le\delta\alpha\}} f.$$

Then

$$\|f_0^\alpha\|_{L^{p_0}}^{p_0} \le (\delta\alpha)^{p_0-p} \|f\|_{L^p}^p, \quad \|f_1^\alpha\|_{L^{p_1}}^{p_1} \le (\delta\alpha)^{p_1-p} \|f\|_{L^p}^p.$$

Quasilinearity of T yields $|T(f)| \le K\left(|T(f_0^\alpha)| + |T(f_1^\alpha)|\right)$ and then

$$\{|T(f)| > \alpha\} \subseteq \{|T(f_0^\alpha)| > \alpha/(2K)\} \cup \{|T(f_1^\alpha)| > \alpha/(2K)\}.$$

It follows that

$$d_{T(f)}(\alpha) \le d_{T(f_0^\alpha)}(\alpha/(2K)) + d_{T(f_1^\alpha)}(\alpha/(2K)).$$

Proof, cont.

We further obtain

$$d_{T(f)}(\alpha) \le \frac{M_0^{p_0}}{(\alpha/(2K))^{p_0}} \int_{\{|f|>\delta\alpha\}} |f|^{p_0}\,\mathrm{d}\mu + \frac{M_1^{p_1}}{(\alpha/(2K))^{p_1}} \int_{\{|f|\le\delta\alpha\}} |f|^{p_1}\,\mathrm{d}\mu.$$

Hence,

$$\begin{aligned}
\|T(f)\|_{L^p}^p \le p(2M_0K)^{p_0} \int_0^\infty \alpha^{p-p_0-1} \int_{\{|f|>\delta\alpha\}} |f|^{p_0}\,\mathrm{d}\mu\mathrm{d}\alpha \\
+ p(2M_1K)^{p_1} \int_0^\infty \alpha^{p-p_1-1} \int_{\{|f|\le\delta\alpha\}} |f|^{p_1}\,\mathrm{d}\mu\mathrm{d}\alpha \\
= p(2M_0K)^{p_0} \int_X |f|^{p_0} \int_0^{|f|/\delta} \alpha^{p-p_0-1}\,\mathrm{d}\alpha\mathrm{d}\mu + p(2M_1K)^{p_1} \int_X |f|^{p_1} \int_{|f|/\delta}^\infty \alpha^{p-p_1-1}\,\mathrm{d}\alpha\mathrm{d}\mu \\
= \left(\frac{(2M_0K)^{p_0}}{p-p_0} \frac{1}{\delta^{p-p_0}} + \frac{(2M_1K)^{p_1}}{p_1-p} \delta^{p_1-p} \right) \|f\|_{L^p}^p.
\end{aligned}$$

The result follows by choosing $\delta > 0$ in such a way that $\frac{M_0^{p_0}}{\delta^{p-p_0}} = M_1^{p_1}\delta^{p_1-p} = M_0^{(1-\theta)p}M_1^{\theta p}$.

Applications

1. The Hilbert transform $H\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is continuous for any $1 < p < \infty$. Here, $Hf(x) = \frac{1}{\pi}\,\mathrm{p.\,v.}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}\,dy$ is convolution with $\mathrm{p.\,v.}\,\frac{1}{\pi x}$.
   - H is unitary on $L^2(\mathbb{R})$ as the Fourier transform of $\mathrm{p.\,v.}\,\frac{1}{\pi x}$ equals $-\mathrm{i}\,\mathrm{sgn}\,\xi$.
   - Then one shows that H is of weak type $(1, 1)$.

   Hence, one gets the result for $1 < p \leq 2$ by interpolation and for $2 \leq p < \infty$ by duality.
2. The Hardy-Littlewood maximal operator M is of weak type $(1, 1)$ and bounded on $L^\infty(\mathbb{R}^n)$, hence bounded on $L^p(\mathbb{R}^n)$ for any $1 < p \leq \infty$ by interpolation. Here, $Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,dy$ for $x \in \mathbb{R}^n$. Note that M is sublinear.

Non-increasing rearrangement

Definition

For $f\colon X \to \mathbb{C}$, the non-increasing rearrangement f^* is defined by

$$f^*(t) = \inf\{s > 0 \mid d_f(s) \le t\}, \quad t > 0,$$

(with the convention that $\inf \emptyset = \infty$).

f^* has the same distribution as f, i.e., $d_f = f_{f*}$. In particular, for any $0 < p < \infty$,

$$\int_X |f|^p \, \mathrm{d}\mu = \int_0^\infty (f^*)^p \, \mathrm{d}t.$$

Lorentz spaces

Definition

For $p, q \in (0, \infty]$, we set

$$\|f\|_{L^{p,q}} = \begin{cases} \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t}\right)^{1/q}, & q < \infty, \\ \sup_{t>0} t^{1/p} f^*(t), & q = \infty, \end{cases}$$

and define

$$L^{p,q}(X, \mu) = \{[f] \mid \|f\|_{p,q} < \infty\}.$$

- $L^{p,q}(X, \mu)$ is a quasi-Banach space. It is a Banach space if $p > 1$, $q \geq 1$ (under an equivalent norm).
- $L^{p,p}(X, \mu) = L^p(X, \mu)$, $L^{p,\infty}(X, \mu)$ is the weak L^p space as previously defined.
- $L^{p,q}(X, \mu) \subseteq L^{p,r}(X, \mu)$ if $q \leq r$.
- $L^{\infty,q}(X, \mu) = \{0\}$ if $q < \infty$.

Lemma

For $p < \infty$,

$$\|f\|_{L^{p,q}} = p^{1/q} \left(\int_0^\infty \left(s d_f(s)^{1/p} \right)^q \frac{ds}{s} \right)^{1/q}.$$

Proof.

Let $q < \infty$. Then the simple functions are dense in $L^{p,q}(X, \mu)$.
For $f(x) = \sum_{k=1}^N a_k \chi_{A_k}(x)$ with the A_k being of finite measure and pairwise disjoint and $a_1 > \ldots > a_N > 0$, one computes

$$d_f(\alpha) = \sum_{j=0}^N b_j \, \chi_{[a_{j+1}, a_j)}(\alpha),$$

where $b_j = \sum_{k \le j} \mu(A_k)$ and $a_0 = \infty$, $a_{N+1} = 0$, as well as

$$\|f\|_{L^{p,q}} = (p/q)^{1/q} \left(a_1^q b_1^{q/p} + a_2^q \left(b_2^{q/p} - b_1^{q/p} \right) + \ldots + a_N^q \left(b_N^{q/p} - b_{N-1}^{q/p} \right) \right)^{1/q}.$$

This shows that the formula above holds in this case. □

Theorem

Let $p_0, p_1, q_0, q_1, r \in (0, \infty]$, $p_0 \neq p_1$, $q_0 \neq q_1$. Let T be quasilinear with domain containing $L^{p_0}(X,\mu) + L^{p_1}(X,\mu)$ or linear with domain containing the simple functions. Suppose that, for all A with finite measure,

$$\|T(\chi_A)\|_{L^{q_j,\infty}} \le M_j\, \mu(A)^{1/p_j}, \quad j = 0, 1.$$

Then, for $\theta \in (0, 1)$,

$$\|T(f)\|_{L^{q_\theta,r}} \le M_{\theta,r}\, \|f\|_{L^{p_\theta,r}}.$$

Note that $L^{p_\theta,\infty}(X,\mu) \subseteq L^{p_0}(X,\mu) + L^{p_1}(X,\mu)$.

Corollary

In addition, suppose that $p_\theta \le q_\theta$. Then

$$\|T(f)\|_{L^{q_\theta}} \le M_\theta\, \|f\|_{L^{p_\theta}}.$$

1 Real interpolation

2 Complex interpolation

The Stein interpolation theorem

Let (X,μ) and (Y,ν) be σ-finite. Further let $p_0, p_1, q_0, q_1 \in [1,\infty]$. Set $S = \{z \in \mathbb{C} \mid 0 < \Re z < 1\}$ and $\overline{S} = \{z \in \mathbb{C} \mid 0 \le \Re z \le 1\}$.

Let $\{T_z\}_{z\in\overline{S}}$ be a family of linear operators which take simple functions on (X,μ) to measurable functions on (Y,ν). Suppose that the following conditions are met:

- $z \mapsto \int_Y (T_z f) g \, \mathrm{d}\nu$ is continuous on $\overline{S}$ and holomorphic on S for all simple functions f, g,
- $\sup_{z\in\overline{S}} \mathrm{e}^{-k|\Im z|} \log \left| \int_Y (T_z f) g \, \mathrm{d}\nu \right| < \infty$ for some $k < \pi$ and all f, g,
- $\sup_{\Re z = j} \mathrm{e}^{-k|\Im z|} \log \| T_z \|_{L^{p_j} \to L^{q_j}} < \infty$ for $j = 0, 1$.

Theorem

Under these conditions, for any $\theta \in (0,1)$, there exists a constant C_θ such that

$$\| T_\theta f \|_{L^{q_\theta}} \le C_\theta \| f \|_{L^{p_\theta}},$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and similar for q_θ.

The Riesz-Thorin interpolation theorem

A special case is when $T_z = T$ is independent of $z \in \overline{S}$.

Theorem

For any $\theta \in (0, 1)$*,*

$$\|T\|_{L^{p_\theta} \to L^{q_\theta}} \leq \|T\|^{1-\theta}_{L^{p_0} \to L^{q_0}} \|T\|^{\theta}_{L^{p_1} \to L^{q_1}} .$$