Topics from harmonic analysis related to generalized Poincaré-Sobolev inequalities: Lecture I

Carlos Pérez

University of the Basque Country
&
BCAM

Summer School on
Dyadic Harmonic Analysis, Martingales, and Paraproducts

Bazaleti, Georgia
September 2-6, 2019

The basic object: the (mean) oscillation of a function, beyond smoothness

The basic object: the (mean) oscillation of a function, beyond smoothness

$$|f(x) - f(y)| \leq c|x-y|^{\alpha} \qquad x, y \in \mathbb{R}^n$$

The basic object: the (mean) oscillation of a function, beyond smoothness

$$|f(x) - f(y)| \leq c|x-y|^{\alpha} \qquad x, y \in \mathbb{R}^n$$

if and only if

The basic object: the (mean) oscillation of a function, beyond smoothness

$$|f(x) - f(y)| \leq c|x-y|^{\alpha} \qquad x, y \in \mathbb{R}^n$$

if and only if

$$\frac{1}{|Q|}\int_Q |f(y) - f_Q|\, dy \leq c\, \ell(Q)^{\alpha}$$

The basic object: the (mean) oscillation of a function, beyond smoothness

$$|f(x)-f(y)| \leq c|x-y|^{\alpha} \qquad x,y\in\mathbb{R}^n$$

if and only if

$$\frac{1}{|Q|}\int_Q |f(y)-f_Q|\,dy \leq c\,\ell(Q)^{\alpha}$$

(Notation: $f_Q := \frac{1}{|Q|}\int_Q f(y)\,dy := ⨍_Q f \qquad (=\langle f\rangle_Q)$)

The basic object: the (mean) oscillation of a function, beyond smoothness

$$|f(x)-f(y)| \leq c|x-y|^{\alpha} \qquad x,y \in \mathbb{R}^n$$

if and only if

$$\frac{1}{|Q|}\int_Q |f(y)-f_Q|\,dy \leq c\,\ell(Q)^{\alpha}$$

(Notation: $f_Q := \frac{1}{|Q|}\int_Q f(y)\,dy := ⨍_Q f \qquad (=< f >_Q)$)

- Due to **Campanato** (early 60's)

The basic object: the (mean) oscillation of a function, beyond smoothness

$$|f(x) - f(y)| \leq c|x-y|^\alpha \qquad x, y \in \mathbb{R}^n$$

if and only if

$$\frac{1}{|Q|}\int_Q |f(y) - f_Q|\, dy \leq c\,\ell(Q)^\alpha$$

(Notation: $f_Q := \frac{1}{|Q|}\int_Q f(y)\,dy := \fint_Q f \qquad (=< f >_Q)$)

- Due to **Campanato** (early 60's)
- The proof is based on the **Lebesgue differentiation theorem**

$$f(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy$$

The basic object: the (mean) oscillation of a function, beyond smoothness

$$|f(x) - f(y)| \leq c|x-y|^{\alpha} \qquad x, y \in \mathbb{R}^n$$

if and only if

$$\frac{1}{|Q|}\int_Q |f(y) - f_Q|\, dy \leq c\,\ell(Q)^{\alpha}$$

(Notation: $f_Q := \frac{1}{|Q|}\int_Q f(y)\, dy := ⨍_Q f \qquad (=< f >_Q)$)

- Due to **Campanato** (early 60's)
- The proof is based on the **Lebesgue differentiation theorem**

$$f(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\, dy$$

- The limiting case, $\alpha = 0$

The basic object: the (mean) oscillation of a function, beyond smoothness

$$|f(x) - f(y)| \leq c|x-y|^{\alpha} \qquad x, y \in \mathbb{R}^n$$

if and only if

$$\frac{1}{|Q|}\int_Q |f(y) - f_Q|\, dy \leq c\, \ell(Q)^{\alpha}$$

(Notation: $f_Q := \frac{1}{|Q|}\int_Q f(y)\, dy := \fint_Q f \qquad (=< f >_Q)$)

- Due to **Campanato** (early 60's)
- The proof is based on the **Lebesgue differentiation theorem**

$$f(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\, dy$$

- The limiting case, $\alpha = 0$ $\qquad \frac{1}{|Q|}\int_Q |f(y) - f_Q|\, dy \leq c$

The basic object: the (mean) oscillation of a function, beyond smoothness

$$|f(x) - f(y)| \leq c|x-y|^\alpha \qquad x, y \in \mathbb{R}^n$$

if and only if

$$\frac{1}{|Q|}\int_Q |f(y) - f_Q|\, dy \leq c\, \ell(Q)^\alpha$$

(Notation: $f_Q := \frac{1}{|Q|}\int_Q f(y)\, dy := \fint_Q f \qquad (=< f >_Q)$)

- Due to **Campanato** (early 60's)
- The proof is based on the **Lebesgue differentiation theorem**

$$f(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\, dy$$

- The limiting case, $\alpha = 0$ $\qquad \frac{1}{|Q|}\int_Q |f(y) - f_Q|\, dy \leq c$

BMO space of **John–Nirenberg**

The basic object: the (mean) oscillation of a function, beyond smoothness

$$|f(x) - f(y)| \leq c|x-y|^\alpha \qquad x, y \in \mathbb{R}^n$$

if and only if

$$\frac{1}{|Q|}\int_Q |f(y) - f_Q|\, dy \leq c\,\ell(Q)^\alpha$$

(Notation: $f_Q := \frac{1}{|Q|}\int_Q f(y)\, dy := \fint_Q f \qquad (=< f >_Q)$)

- Due to **Campanato** (early 60's)
- The proof is based on the **Lebesgue differentiation theorem**

$$f(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\, dy$$

- The limiting case, $\alpha = 0$ $\qquad \frac{1}{|Q|}\int_Q |f(y) - f_Q|\, dy \leq c$

BMO space of **John–Nirenberg**

- Examples: $f(x) = \log|x|$

The basic object: the (mean) oscillation of a function, beyond smoothness

$$|f(x) - f(y)| \leq c|x-y|^\alpha \qquad x, y \in \mathbb{R}^n$$

if and only if

$$\frac{1}{|Q|}\int_Q |f(y) - f_Q|\,dy \leq c\,\ell(Q)^\alpha$$

(Notation: $f_Q := \frac{1}{|Q|}\int_Q f(y)\,dy := \fint_Q f \qquad (=< f >_Q)$)

- Due to **Campanato** (early 60's)
- The proof is based on the **Lebesgue differentiation theorem**

$$f(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy$$

- The limiting case, $\alpha = 0$ $\qquad \frac{1}{|Q|}\int_Q |f(y) - f_Q|\,dy \leq c$

BMO space of **John–Nirenberg**

- Examples: $f(x) = \log|x|$ or more generally $f(x) = \log M(\mu)(x)$

One first key operator

One first key operator

The (modern) proof of the Lebesgue differentiation theorem is based on certain properties of

One first key operator

The (modern) proof of the Lebesgue differentiation theorem is based on certain properties of the **Hardy–Littlewood maximal function**

One first key operator

The (modern) proof of the Lebesgue differentiation theorem is based on certain properties of the **Hardy–Littlewood maximal function**

$$Mf(x) := \sup_{x\in Q} \fint_Q |f(y)|\, dy$$

One first key operator

The (modern) proof of the Lebesgue differentiation theorem is based on certain properties of the **Hardy–Littlewood maximal function**

$$Mf(x) := \sup_{x \in Q} \fint_Q |f(y)| \, dy \approx \sup_{r>0} \fint_{B_r(x)} |f(y)| \, dy$$

One first key operator

The (modern) proof of the Lebesgue differentiation theorem is based on certain properties of the **Hardy–Littlewood maximal function**

$$Mf(x) := \sup_{x\in Q} \fint_Q |f(y)|\, dy \approx \sup_{r>0} \fint_{B_r(x)} |f(y)|\, dy$$

Classical results:

One first key operator

The (modern) proof of the Lebesgue differentiation theorem is based on certain properties of the **Hardy–Littlewood maximal function**

$$Mf(x) := \sup_{x\in Q} \fint_Q |f(y)|\,dy \approx \sup_{r>0} \fint_{B_r(x)} |f(y)|\,dy$$

Classical results:

$$M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad 1 < p \leq \infty$$

and specially

One first key operator

The (modern) proof of the Lebesgue differentiation theorem is based on certain properties of the **Hardy–Littlewood maximal function**

$$Mf(x) := \sup_{x\in Q} \fint_Q |f(y)|\,dy \approx \sup_{r>0} \fint_{B_r(x)} |f(y)|\,dy$$

Classical results:

$$M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad 1 < p \le \infty$$

and specially

$$M : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$$

namely

One first key operator

The (modern) proof of the Lebesgue differentiation theorem is based on certain properties of the **Hardy–Littlewood maximal function**

$$Mf(x) := \sup_{x\in Q} \fint_Q |f(y)|\,dy \approx \sup_{r>0} \fint_{B_r(x)} |f(y)|\,dy$$

Classical results:

$$M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad 1 < p \leq \infty$$

and specially

$$M : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$$

namely

$$\|Mf\|_{L^{1,\infty}(\mathbb{R}^n)} \leq c_n \int_{\mathbb{R}^n} |f(x)|\,dx$$

One first key operator

The (modern) proof of the Lebesgue differentiation theorem is based on certain properties of the **Hardy–Littlewood maximal function**

$$Mf(x) := \sup_{x\in Q} \fint_Q |f(y)|\,dy \approx \sup_{r>0} \fint_{B_r(x)} |f(y)|\,dy$$

Classical results:

$$M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad 1 < p \leq \infty$$

and specially

$$M : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$$

namely

$$\|Mf\|_{L^{1,\infty}(\mathbb{R}^n)} \leq c_n \int_{\mathbb{R}^n} |f(x)|\,dx$$

where

$$\|f\|_{L^{p,\infty}_\mu} := \sup_{\lambda>0} \lambda \mu(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})^{\frac{1}{p}}$$

One first key operator

The (modern) proof of the Lebesgue differentiation theorem is based on certain properties of the **Hardy–Littlewood maximal function**

$$Mf(x) := \sup_{x\in Q} ⨍_Q |f(y)|\, dy \approx \sup_{r>0} ⨍_{B_r(x)} |f(y)|\, dy$$

Classical results:

$$M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad 1 < p \leq \infty$$

and specially

$$M : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$$

namely

$$\|Mf\|_{L^{1,\infty}(\mathbb{R}^n)} \leq c_n \int_{\mathbb{R}^n} |f(x)|\, dx$$

where

$$\|f\|_{L^{p,\infty}_\mu} := \sup_{\lambda>0} \lambda \mu(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})^{\frac{1}{p}}$$

Somehow this is a ”rough” estimate of f because we have to measure

One first key operator

The (modern) proof of the Lebesgue differentiation theorem is based on certain properties of the **Hardy–Littlewood maximal function**

$$Mf(x) := \sup_{x\in Q} \fint_Q |f(y)|\,dy \approx \sup_{r>0} \fint_{B_r(x)} |f(y)|\,dy$$

Classical results:

$$M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad 1 < p \leq \infty$$

and specially

$$M : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$$

namely

$$\|Mf\|_{L^{1,\infty}(\mathbb{R}^n)} \leq c_n \int_{\mathbb{R}^n} |f(x)|\,dx$$

where

$$\|f\|_{L^{p,\infty}_\mu} := \sup_{\lambda>0} \lambda \mu(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})^{\frac{1}{p}}$$

Somehow this is a ”rough” estimate of f because we have to measure

$$|f(x) - f_{B(x,r)}| = \left| f(x) - \fint_{B(x,r)} f(y)dy \right|$$

The basic object: the oscillation of a function, beyond smoothness II

The basic object: the oscillation of a function, beyond smoothness II

Motivated by the Variance of a random variable X

The basic object: the oscillation of a function, beyond smoothness II

Motivated by the Variance of a random variable X

$$\fint_Q |f(y) - f_Q| \, dy$$

The basic object: the oscillation of a function, beyond smoothness II

Motivated by the Variance of a random variable X

$$\fint_Q |f(y) - f_Q|\, dy$$

$$\approx \inf_c \fint_Q |f(y) - c|\, dy$$

The basic object: the oscillation of a function, beyond smoothness II

Motivated by the Variance of a random variable X

$$\fint_Q |f(y) - f_Q|\,dy$$

$$\approx \inf_c \fint_Q |f(y) - c|\,dy$$

$$\approx \fint_Q \fint_Q |f(x) - f(y)|\,dxdy$$

The basic object: the oscillation of a function, beyond smoothness II

Motivated by the Variance of a random variable X

$$\fint_Q |f(y) - f_Q| \, dy$$

$$\approx \inf_c \fint_Q |f(y) - c| \, dy$$

$$\approx \fint_Q \fint_Q |f(x) - f(y)| \, dx dy$$

Possible variations:

The basic object: the oscillation of a function, beyond smoothness II

Motivated by the Variance of a random variable X

$$\fint_Q |f(y) - f_Q| \, dy$$

$$\approx \inf_c \fint_Q |f(y) - c| \, dy$$

$$\approx \fint_Q \fint_Q |f(x) - f(y)| \, dx dy$$

Possible variations:

$$\left(\fint_Q |f(y) - f_Q|^{\delta} \, dy \right)^{\frac{1}{\delta}}$$

The basic object: the oscillation of a function, beyond smoothness II

Motivated by the Variance of a random variable X

$$\fint_Q |f(y) - f_Q| \, dy$$

$$\approx \inf_c \fint_Q |f(y) - c| \, dy$$

$$\approx \fint_Q \fint_Q |f(x) - f(y)| \, dx dy$$

Possible variations:

$$\left(\fint_Q |f(y) - f_Q|^\delta \, dy \right)^{\frac{1}{\delta}}$$

or other ”norms”. (The case $\delta = 2$ is the variance but we think more in $\delta < 1$.)

The basic object: the oscillation of a function, beyond smoothness II

Motivated by the Variance of a random variable X

$$\fint_Q |f(y) - f_Q|\, dy$$

$$\approx \inf_c \fint_Q |f(y) - c|\, dy$$

$$\approx \fint_Q \fint_Q |f(x) - f(y)|\, dx dy$$

Possible variations:

$$\left(\fint_Q |f(y) - f_Q|^\delta\, dy \right)^{\frac{1}{\delta}}$$

or other ”norms”. (The case $\delta = 2$ is the variance but we think more in $\delta < 1$.)

Another variation,

The basic object: the oscillation of a function, beyond smoothness II

Motivated by the Variance of a random variable X

$$\fint_Q |f(y) - f_Q|\, dy$$

$$\approx \inf_c \fint_Q |f(y) - c|\, dy$$

$$\approx \fint_Q \fint_Q |f(x) - f(y)|\, dxdy$$

Possible variations:

$$\left(\fint_Q |f(y) - f_Q|^\delta\, dy\right)^{\frac{1}{\delta}}$$

or other ”norms”. (The case $\delta = 2$ is the variance but we think more in $\delta < 1$.)

Another variation, $\inf_{\pi\in\mathcal{P}_k} \frac{1}{|Q|} \int_Q |f - \pi|$

The basic object: the oscillation of a function, beyond smoothness II

Motivated by the Variance of a random variable X

$$\fint_Q |f(y) - f_Q|\, dy$$

$$\approx \inf_c \fint_Q |f(y) - c|\, dy$$

$$\approx \fint_Q \fint_Q |f(x) - f(y)|\, dx dy$$

Possible variations:

$$\left(\fint_Q |f(y) - f_Q|^\delta\, dy \right)^{\frac{1}{\delta}}$$

or other "norms". (The case $\delta = 2$ is the variance but we think more in $\delta < 1$.)

Another variation, $\inf_{\pi \in \mathcal{P}_k} \frac{1}{|Q|} \int_Q |f - \pi| \approx \frac{1}{|Q|} \int_Q |f - P_Q(f)|$

The basic object: the oscillation of a function, beyond smoothness II

Motivated by the Variance of a random variable X

$$\fint_Q |f(y) - f_Q|\, dy$$

$$\approx \inf_c \fint_Q |f(y) - c|\, dy$$

$$\approx \fint_Q \fint_Q |f(x) - f(y)|\, dx dy$$

Possible variations:

$$\left(\fint_Q |f(y) - f_Q|^{\delta}\, dy\right)^{\frac{1}{\delta}}$$

or other "norms". (The case $\delta = 2$ is the variance but we think more in $\delta < 1$.)

Another variation, $\inf_{\pi \in \mathcal{P}_k} \frac{1}{|Q|} \int_Q |f - \pi| \approx \frac{1}{|Q|} \int_Q |f - P_Q(f)|$

where $P_Q(f)$ is a unique polynomial satisfying certain properties.

The basic object: the oscillation of a function, beyond smoothness II

Motivated by the Variance of a random variable X

$$\fint_Q |f(y) - f_Q|\, dy$$

$$\approx \inf_c \fint_Q |f(y) - c|\, dy$$

$$\approx \fint_Q \fint_Q |f(x) - f(y)|\, dx dy$$

Possible variations:

$$\left(\fint_Q |f(y) - f_Q|^\delta\, dy \right)^{\frac{1}{\delta}}$$

or other "norms". (The case $\delta = 2$ is the variance but we think more in $\delta < 1$.)

Another variation, $\inf_{\pi \in \mathcal{P}_k} \frac{1}{|Q|} \int_Q |f - \pi| \approx \frac{1}{|Q|} \int_Q |f - P_Q(f)|$

where $P_Q(f)$ is a unique polynomial satisfying certain properties.

Resembling Taylor polynomials.

THE CLASSICAL POINCARÉ INEQUALITY

THE CLASSICAL POINCARÉ INEQUALITY

- $(1,1)$ **Poincaré inequality:**

$$\frac{1}{|Q|}\int_Q |f-f_Q| \leq c_n \, \frac{\ell(Q)}{|Q|}\int_Q |\nabla f|$$

THE CLASSICAL POINCARÉ INEQUALITY

- $(1,1)$ **Poincaré inequality:**

$$\frac{1}{|Q|}\int_Q |f - f_Q| \leq c_n \frac{\ell(Q)}{|Q|}\int_Q |\nabla f|$$

- The (p,p) version, $p \geq 1$:

$$\left(\frac{1}{|Q|}\int_Q |f - f_Q|^p\, dx\right)^{1/p} \leq c\,\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f|^p dx\right)^{1/p}$$

THE CLASSICAL POINCARÉ INEQUALITY

- $(1,1)$ **Poincaré inequality:**

$$\frac{1}{|Q|}\int_Q |f-f_Q| \le c_n\,\frac{\ell(Q)}{|Q|}\int_Q |\nabla f|$$

- The (p,p) version, $p \geq 1$:

$$\left(\frac{1}{|Q|}\int_Q |f-f_Q|^p\,dx\right)^{1/p} \le c\,\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f|^p dx\right)^{1/p}$$

- **Higher order case:**

THE CLASSICAL POINCARÉ INEQUALITY

- $(1,1)$ **Poincaré inequality:**

$$\frac{1}{|Q|}\int_Q |f-f_Q| \leq c_n \frac{\ell(Q)}{|Q|}\int_Q |\nabla f|$$

- The (p,p) version, $p \geq 1$:

$$\left(\frac{1}{|Q|}\int_Q |f-f_Q|^p\,dx\right)^{1/p} \leq c\,\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f|^p dx\right)^{1/p}$$

- **Higher order case:**

$$\frac{1}{|Q|}\int_Q |f-P_Q(f)| \leq c_n \frac{\ell(Q)^m}{|Q|}\int_Q |\nabla^m f| \qquad m=1,2\cdots$$

THE CLASSICAL POINCARÉ INEQUALITY

- $(1,1)$ **Poincaré inequality:**

$$\frac{1}{|Q|}\int_Q |f-f_Q| \le c_n\,\frac{\ell(Q)}{|Q|}\int_Q |\nabla f|$$

- The (p,p) version, $p \ge 1$:

$$\left(\frac{1}{|Q|}\int_Q |f-f_Q|^p\,dx\right)^{1/p} \le c\,\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f|^p dx\right)^{1/p}$$

- **Higher order case:**

$$\frac{1}{|Q|}\int_Q |f-P_Q(f)| \le c_n\,\frac{\ell(Q)^m}{|Q|}\int_Q |\nabla^m f| \qquad m=1,2\cdots$$

where $P_Q(f)$ is an appropriate optimal polynomial of order $m-1$ in the sense that

THE CLASSICAL POINCARÉ INEQUALITY

- $(1,1)$ **Poincaré inequality:**

$$\frac{1}{|Q|}\int_Q |f - f_Q| \leq c_n \frac{\ell(Q)}{|Q|}\int_Q |\nabla f|$$

- The (p,p) version, $p \geq 1$:

$$\left(\frac{1}{|Q|}\int_Q |f - f_Q|^p \, dx\right)^{1/p} \leq c\,\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f|^p dx\right)^{1/p}$$

- **Higher order case:**

$$\frac{1}{|Q|}\int_Q |f - P_Q(f)| \leq c_n \frac{\ell(Q)^m}{|Q|}\int_Q |\nabla^m f| \qquad m = 1, 2 \cdots$$

where $P_Q(f)$ is an appropriate optimal polynomial of order $m-1$ in the sense that

$$\inf_{\pi \in \mathcal{P}} \frac{1}{|Q|}\int_Q |f - \pi| \approx \frac{1}{|Q|}\int_Q |f - P_Q(f)|$$

INTEGRAL REPRESENTATION FORMULA in $\mathbb{R}^n$

INTEGRAL REPRESENTATION FORMULA in $\mathbb{R}^n$

- **Fractional Integral Operator:**

INTEGRAL REPRESENTATION FORMULA in $\mathbb{R}^n$

- **Fractional Integral Operator:** If $\alpha \in (0, n)$

INTEGRAL REPRESENTATION FORMULA in $\mathbb{R}^n$

- **Fractional Integral Operator:** If $\alpha \in (0, n)$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy = (K_\alpha * f)(x)$$

with $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$

INTEGRAL REPRESENTATION FORMULA in $\mathbb{R}^n$

- **Fractional Integral Operator:** If $\alpha \in (0, n)$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy = (K_\alpha * f)(x)$$

with $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$

Lemma Let $f \in C^1(Q)$, then

$$|f(x) - f_Q| \leq c\, I_1(|\nabla f| \chi_Q)(x) \qquad x \in Q$$

INTEGRAL REPRESENTATION FORMULA in $\mathbb{R}^n$

- **Fractional Integral Operator:** If $\alpha \in (0, n)$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy = (K_\alpha * f)(x)$$

with $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$

Lemma Let $f \in C^1(Q)$, then

$$|f(x) - f_Q| \leq c\, I_1(|\nabla f|\chi_Q)(x) \qquad x \in Q$$

- It goes back to **Sobolev**

INTEGRAL REPRESENTATION FORMULA in $\mathbb{R}^n$

- **Fractional Integral Operator:** If $\alpha \in (0, n)$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy = (K_\alpha * f)(x)$$

with $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$

Lemma Let $f \in C^1(Q)$, then

$$|f(x) - f_Q| \leq c\, I_1(|\nabla f| \chi_Q)(x) \qquad x \in Q$$

- It goes back to **Sobolev**

Proof

INTEGRAL REPRESENTATION FORMULA in $\mathbb{R}^n$

• **Fractional Integral Operator:** If $\alpha \in (0,n)$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy = (K_\alpha * f)(x)$$

with $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$

Lemma Let $f \in C^1(Q)$, then

$$|f(x) - f_Q| \leq c\, I_1(|\nabla f|\chi_Q)(x) \qquad x \in Q$$

• It goes back to **Sobolev**

Proof If $x \in Q$, $|f(x) - f_Q| = \left| f(x) - ⨍_Q f(y)dy \right| \leq ⨍_Q |f(x) - f(y)|dy$

INTEGRAL REPRESENTATION FORMULA in $\mathbb{R}^n$

• **Fractional Integral Operator:** If $\alpha \in (0,n)$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy = (K_\alpha * f)(x)$$

with $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$

Lemma Let $f \in C^1(Q)$, then

$$|f(x) - f_Q| \leq c\, I_1(|\nabla f|\chi_Q)(x) \qquad x \in Q$$

• It goes back to **Sobolev**

Proof If $x \in Q$, $|f(x) - f_Q| = \left| f(x) - \fint_Q f(y)dy \right| \leq \fint_Q |f(x) - f(y)|dy$

We define

INTEGRAL REPRESENTATION FORMULA in $\mathbb{R}^n$

• **Fractional Integral Operator:** If $\alpha \in (0, n)$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy = (K_\alpha * f)(x)$$

with $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$

Lemma Let $f \in C^1(Q)$, then

$$|f(x) - f_Q| \leq c\, I_1(|\nabla f|\chi_Q)(x) \qquad x \in Q$$

• It goes back to **Sobolev**

Proof If $x \in Q$, $|f(x) - f_Q| = \left| f(x) - \fint_Q f(y)dy \right| \leq \fint_Q |f(x) - f(y)|dy$

We define

$$\gamma(t) := f(x + t(y-x))$$

INTEGRAL REPRESENTATION FORMULA in $\mathbb{R}^n$

• **Fractional Integral Operator:** If $\alpha \in (0, n)$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy = (K_\alpha * f)(x)$$

with $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$

Lemma Let $f \in C^1(Q)$, then

$$|f(x) - f_Q| \leq c\, I_1(|\nabla f| \chi_Q)(x) \qquad x \in Q$$

• It goes back to **Sobolev**

Proof If $x \in Q$, $|f(x) - f_Q| = \left| f(x) - \fint_Q f(y)dy \right| \leq \fint_Q |f(x) - f(y)| dy$

We define

$$\gamma(t) := f(x + t(y - x))$$

and observe

$$\gamma'(t) = \nabla f(x + t(y - x)).(y - x)$$

INTEGRAL REPRESENTATION FORMULA in $\mathbb{R}^n$

- **Fractional Integral Operator:** If $\alpha \in (0, n)$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy = (K_\alpha * f)(x)$$

with $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$

Lemma Let $f \in C^1(Q)$, then

$$|f(x) - f_Q| \leq c\, I_1(|\nabla f|\chi_Q)(x) \qquad x \in Q$$

- It goes back to **Sobolev**

Proof If $x \in Q$, $|f(x) - f_Q| = \left| f(x) - \fint_Q f(y)dy \right| \leq \fint_Q |f(x) - f(y)| dy$

We define

$$\gamma(t) := f(x + t(y-x))$$

and observe

$$\gamma'(t) = \nabla f(x + t(y-x)).(y-x)$$

so that

INTEGRAL REPRESENTATION FORMULA in $\mathbb{R}^n$

• **Fractional Integral Operator:** If $\alpha \in (0, n)$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy = (K_\alpha * f)(x)$$

with $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$

Lemma Let $f \in C^1(Q)$, then

$$|f(x) - f_Q| \leq c\, I_1(|\nabla f|\chi_Q)(x) \qquad x \in Q$$

• It goes back to **Sobolev**

Proof If $x \in Q$, $|f(x) - f_Q| = \left| f(x) - ⨍_Q f(y)dy \right| \leq ⨍_Q |f(x) - f(y)|dy$

We define

$$\gamma(t) := f(x + t(y-x))$$

and observe

$$\gamma'(t) = \nabla f(x + t(y-x)).(y-x)$$

so that

$$f(y) - f(x) = \gamma(1) - \gamma(0) = \int_0^1 \gamma'(t)dt = \int_0^1 \nabla f(x + t(y-x)).(y-x)dt$$

INTEGRAL REPRESENTATION FORMULA II

INTEGRAL REPRESENTATION FORMULA II

Hence, integration we have

INTEGRAL REPRESENTATION FORMULA II

Hence, integration we have

$$\fint_Q |f(x)-f(y)|dy \leq \fint_Q \left| \int_0^1 \nabla f(x+t(y-x))(y-x)dt \right| dy$$
$$\leq \fint_Q \int_0^1 |\nabla f(x+t(y-x))|\,|y-x|\,dtdy$$
$$\leq \frac{1}{|Q|} \int_0^1 \int_{B(x,\sqrt{n}l(Q))} |\nabla f(x+t(y-x))|\,|y-x|\,\chi_Q(y)dydt.$$

INTEGRAL REPRESENTATION FORMULA II

Hence, integration we have

$$\begin{aligned}
&\fint_Q |f(x) - f(y)| dy \leq \fint_Q \left| \int_0^1 \nabla f(x + t(y-x))(y-x) dt \right| dy \\
&\leq \fint_Q \int_0^1 |\nabla f(x + t(y-x))| \, |y-x| \, dt dy \\
&\leq \frac{1}{|Q|} \int_0^1 \int_{B(x,\sqrt{n}l(Q))} |\nabla f(x + t(y-x))| \, |y-x| \, \chi_Q(y) dy dt.
\end{aligned}$$

Change of variable $z = x + t(y-x)$ combined with Fubini's

INTEGRAL REPRESENTATION FORMULA II

Hence, integration we have

$$
\begin{aligned}
\fint_Q |f(x)-f(y)|dy &\leq \fint_Q \left| \int_0^1 \nabla f(x+t(y-x))(y-x)dt \right| dy \\
&\leq \fint_Q \int_0^1 |\nabla f(x+t(y-x))|\,|y-x|\,dtdy \\
&\leq \frac{1}{|Q|}\int_0^1 \int_{B(x,\sqrt{n}l(Q))} |\nabla f(x+t(y-x))|\,|y-x|\,\chi_Q(y)dydt.
\end{aligned}
$$

Change of variable $z = x + t(y-x)$ combined with Fubini's

$$
\begin{aligned}
&\frac{1}{|Q|}\int_0^1 \int_{B(x,t\sqrt{n}l(Q))} |\nabla f(z)|\,\frac{|z-x|}{t}\frac{1}{t^n}\chi_Q(z)dzdt \\
&= \fint_Q \int_{\frac{|z-x|}{C_n l(Q)}}^{\infty} \frac{1}{t^n}\frac{dt}{t}\,|\nabla f(z)|\,|z-x|\,dz \\
&= C_n \int_Q \frac{|\nabla f(z)|}{|z-x|^{n-1}}dz = C_n I_1(|\nabla f|\chi_Q)(x).
\end{aligned}
$$

□

The simplest Poincaré inequality

The simplest Poincaré inequality

Corollary Poincaré $(1,1)$ inequality:

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \leq C_n\frac{\ell(Q)}{|Q|}\int_Q |\nabla f(x)|\,dx$$

The simplest Poincaré inequality

Corollary Poincaré $(1,1)$ inequality:

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \le C_n \frac{\ell(Q)}{|Q|}\int_Q |\nabla f(x)|\,dx$$

Proof Fubini applied to the pontwise estimate

The simplest Poincaré inequality

Corollary Poincaré $(1,1)$ inequality:

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \le C_n \frac{\ell(Q)}{|Q|}\int_Q |\nabla f(x)|\,dx$$

Proof Fubini applied to the pontwise estimate

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx &\le \frac{c_n}{|Q|}\int_Q I_1\left(|\nabla f|\chi_Q\right)(x)dx\\ &= \frac{c_n}{|Q|}\int_Q \left(\int_Q \frac{g(y)}{|x-y|^{n-1}}dy\right)dx\\ &= \frac{c_n}{|Q|}\int_Q g(y)\left(\int_Q \frac{1}{|x-y|^{n-1}}dx\right)dy\end{aligned}$$

The simplest Poincaré inequality

Corollary Poincaré $(1,1)$ inequality:

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \le C_n \frac{\ell(Q)}{|Q|}\int_Q |\nabla f(x)|\,dx$$

Proof Fubini applied to the pontwise estimate

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx &\le \frac{c_n}{|Q|}\int_Q I_1\left(|\nabla f|\chi_Q\right)(x)dx \\ &= \frac{c_n}{|Q|}\int_Q \left(\int_Q \frac{g(y)}{|x-y|^{n-1}}dy\right)dx \\ &= \frac{c_n}{|Q|}\int_Q g(y)\left(\int_Q \frac{1}{|x-y|^{n-1}}dx\right)dy\end{aligned}$$

To finish we just need to prove

The simplest Poincaré inequality

Corollary Poincaré $(1,1)$ inequality:

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \le C_n \frac{\ell(Q)}{|Q|}\int_Q |\nabla f(x)|\,dx$$

Proof Fubini applied to the pontwise estimate

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx &\le \frac{c_n}{|Q|}\int_Q I_1\left(|\nabla f|\chi_Q\right)(x)dx \\ &= \frac{c_n}{|Q|}\int_Q \left(\int_Q \frac{g(y)}{|x-y|^{n-1}}dy\right)dx \\ &= \frac{c_n}{|Q|}\int_Q g(y)\left(\int_Q \frac{1}{|x-y|^{n-1}}dx\right)dy\end{aligned}$$

To finish we just need to prove

$$\int_Q \frac{1}{|x-y|^{n-1}}dx \le c_n \ell(Q) \qquad x\in\mathbb{R}^n$$

The simplest Poincaré inequality

Corollary Poincaré $(1,1)$ inequality:

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \le C_n \frac{\ell(Q)}{|Q|}\int_Q |\nabla f(x)|\, dx$$

Proof Fubini applied to the pontwise estimate

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx &\le \frac{c_n}{|Q|}\int_Q I_1\left(|\nabla f|\chi_Q\right)(x)dx \\ &= \frac{c_n}{|Q|}\int_Q \left(\int_Q \frac{g(y)}{|x-y|^{n-1}}dy\right)dx \\ &= \frac{c_n}{|Q|}\int_Q g(y)\left(\int_Q \frac{1}{|x-y|^{n-1}}dx\right)dy\end{aligned}$$

To finish we just need to prove

$$\int_Q \frac{1}{|x-y|^{n-1}}dx \le c_n \ell(Q) \qquad x\in\mathbb{R}^n$$

We will prove something a bit more general.

The "layer cake formula"

The “layer cake formula”

The Layer cake formula Let $\phi : [0, \infty) \longrightarrow [0, \infty)$, increasing, C^1, with $\phi(0) = 0$, then

$$\int_X \phi\left(f(x)\right) d\mu = \int_0^\infty \phi'(t)\mu\left(\{x \in X \,:\, f(x) > t\}\right) dt$$

The "layer cake formula"

The Layer cake formula Let $\phi : [0,\infty) \longrightarrow [0,\infty)$, increasing, C^1, with $\phi(0) = 0$, then

$$\int_X \phi\left(f(x)\right) d\mu = \int_0^\infty \phi'(t)\mu\left(\{x \in X \,:\, f(x) > t\}\right) dt$$

Lemma Let $0 < \alpha < n$ and let $\Omega \subset \mathbb{R}^n$ with finite measure. Then

$$\int_\Omega \frac{1}{|x-y|^{n-\alpha}} dx \leq c_n |\Omega|^{\frac{\alpha}{n}} \qquad x \in \mathbb{R}^n$$

The “layer cake formula”

The Layer cake formula Let $\phi : [0,\infty) \longrightarrow [0,\infty)$, increasing, C^1, with $\phi(0) = 0$, then

$$\int_X \phi\left(f(x)\right) d\mu = \int_0^\infty \phi'(t)\mu\left(\{x \in X \,:\, f(x) > t\}\right) dt$$

Lemma Let $0 < \alpha < n$ and let $\Omega \subset \mathbb{R}^n$ with finite measure. Then

$$\int_\Omega \frac{1}{|x-y|^{n-\alpha}} dx \leq c_n |\Omega|^{\frac{\alpha}{n}} \qquad x \in \mathbb{R}^n$$

Proof WMA $x = 0$.

The "layer cake formula"

The Layer cake formula Let $\phi : [0,\infty) \longrightarrow [0,\infty)$, increasing, C^1, with $\phi(0) = 0$, then

$$\int_X \phi\left(f(x)\right) d\mu = \int_0^\infty \phi'(t)\mu\left(\{x \in X \,:\, f(x) > t\}\right) dt$$

Lemma Let $0 < \alpha < n$ and let $\Omega \subset \mathbb{R}^n$ with finite measure. Then

$$\int_\Omega \frac{1}{|x-y|^{n-\alpha}} dx \leq c_n |\Omega|^{\frac{\alpha}{n}} \qquad x \in \mathbb{R}^n$$

Proof WMA $x = 0$. Then

The “layer cake formula”

The Layer cake formula Let $\phi : [0,\infty) \longrightarrow [0,\infty)$, increasing, C^1, with $\phi(0) = 0$, then

$$\int_X \phi\left(f(x)\right) d\mu = \int_0^\infty \phi'(t)\mu\left(\{x \in X \,:\, f(x) > t\}\right) dt$$

Lemma Let $0 < \alpha < n$ and let $\Omega \subset \mathbb{R}^n$ with finite measure. Then

$$\int_\Omega \frac{1}{|x-y|^{n-\alpha}} dx \leq c_n |\Omega|^{\frac{\alpha}{n}} \qquad x \in \mathbb{R}^n$$

Proof WMA $x = 0$. Then

$$\int_\Omega \frac{1}{|y|^{n-\alpha}}\, dy = \int_0^\infty \left|\left\{y \in \Omega : \tfrac{1}{|y|^{n-\alpha}} > t\right\}\right| dt$$

The "layer cake formula"

The Layer cake formula Let $\phi : [0,\infty) \longrightarrow [0,\infty)$, increasing, C^1, with $\phi(0) = 0$, then

$$\int_X \phi\left(f(x)\right) d\mu = \int_0^\infty \phi'(t)\mu\left(\{x \in X \,:\, f(x) > t\}\right) dt$$

Lemma Let $0 < \alpha < n$ and let $\Omega \subset \mathbb{R}^n$ with finite measure. Then

$$\int_\Omega \frac{1}{|x-y|^{n-\alpha}} dx \leq c_n |\Omega|^{\frac{\alpha}{n}} \qquad x \in \mathbb{R}^n$$

Proof WMA $x = 0$. Then

$$\int_\Omega \frac{1}{|y|^{n-\alpha}}\, dy = \int_0^\infty \left|\left\{y \in \Omega : \tfrac{1}{|y|^{n-\alpha}} > t\right\}\right| dt$$

$$= \int_0^\infty \left|\left\{y \in \Omega : |y| < (\tfrac{1}{t})^{\frac{1}{n-\alpha}}\right\}\right| dt$$

The "layer cake formula"

The Layer cake formula Let $\phi : [0,\infty) \longrightarrow [0,\infty)$, increasing, C^1, with $\phi(0) = 0$, then

$$\int_X \phi\left(f(x)\right) d\mu = \int_0^\infty \phi'(t)\mu\left(\{x \in X \,:\, f(x) > t\}\right) dt$$

Lemma Let $0 < \alpha < n$ and let $\Omega \subset \mathbb{R}^n$ with finite measure. Then

$$\int_\Omega \frac{1}{|x-y|^{n-\alpha}} dx \leq c_n |\Omega|^{\frac{\alpha}{n}} \qquad x \in \mathbb{R}^n$$

Proof WMA $x = 0$. Then

$$\int_\Omega \frac{1}{|y|^{n-\alpha}}\, dy = \int_0^\infty \left|\left\{y \in \Omega : \tfrac{1}{|y|^{n-\alpha}} > t\right\}\right| dt$$

$$= \int_0^\infty \left|\left\{y \in \Omega : |y| < (\tfrac{1}{t})^{\frac{1}{n-\alpha}}\right\}\right| dt$$

$$\leq \int_0^\infty \min\{|\Omega|, \frac{c_n}{t^{\frac{n}{n-\alpha}}}\}\, dt \approx |\Omega|^{\frac{\alpha}{n}}$$

□

The "layer cake formula"

The Layer cake formula Let $\phi : [0,\infty) \longrightarrow [0,\infty)$, increasing, C^1, with $\phi(0) = 0$, then

$$\int_X \phi\left(f(x)\right) d\mu = \int_0^\infty \phi'(t)\mu\left(\{x \in X \,:\, f(x) > t\}\right) dt$$

Lemma Let $0 < \alpha < n$ and let $\Omega \subset \mathbb{R}^n$ with finite measure. Then

$$\int_\Omega \frac{1}{|x-y|^{n-\alpha}} dx \le c_n |\Omega|^{\frac{\alpha}{n}} \qquad x \in \mathbb{R}^n$$

Proof WMA $x = 0$. Then

$$\int_\Omega \frac{1}{|y|^{n-\alpha}}\, dy = \int_0^\infty \left|\left\{y \in \Omega : \tfrac{1}{|y|^{n-\alpha}} > t\right\}\right| dt$$

$$= \int_0^\infty \left|\left\{y \in \Omega : |y| < (\tfrac{1}{t})^{\frac{1}{n-\alpha}}\right\}\right| dt$$

$$\le \int_0^\infty \min\{|\Omega|, \frac{c_n}{t^{\frac{n}{n-\alpha}}}\}\, dt \approx |\Omega|^{\frac{\alpha}{n}}$$

□

The "layer cake formula"

The Layer cake formula Let $\phi : [0,\infty) \longrightarrow [0,\infty)$, increasing, C^1, with $\phi(0) = 0$, then

$$\int_X \phi\left(f(x)\right) d\mu = \int_0^\infty \phi'(t)\mu\left(\{x \in X \,:\, f(x) > t\}\right) dt$$

Lemma Let $0 < \alpha < n$ and let $\Omega \subset \mathbb{R}^n$ with finite measure. Then

$$\int_\Omega \frac{1}{|x-y|^{n-\alpha}} dx \le c_n |\Omega|^{\frac{\alpha}{n}} \qquad x \in \mathbb{R}^n$$

Proof WMA $x = 0$. Then

$$\int_\Omega \frac{1}{|y|^{n-\alpha}}\, dy = \int_0^\infty \left|\left\{y \in \Omega : \tfrac{1}{|y|^{n-\alpha}} > t\right\}\right| dt$$

$$= \int_0^\infty \left|\left\{y \in \Omega : |y| < (\tfrac{1}{t})^{\frac{1}{n-\alpha}}\right\}\right| dt$$

$$\le \int_0^\infty \min\{|\Omega|, \frac{c_n}{t^{\frac{n}{n-\alpha}}}\}\, dt \approx |\Omega|^{\frac{\alpha}{n}}$$

□

END FIRST LECTURE